The Calderon-Zygmund Decomposition on Product Domains
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Outline

- Definition of the Hardy Spaces
- Atomic Decomposition on H^1
- Calderon-Zygmund Decomposition
- Atomic Decomposition on H^p

Definition of the Hardy Spaces

The Hardy spaces $H^p(\mathbf{R}^n)$ have several equivalent definitions.

- The oldest is for $n = 1$ the space $H^p(\mathbf{R})$ can be defined as the boundary values on the real axis of the real parts of those analytic functions f in the upper half-plane which satisfy

$$\|f\|_{H^p} = \sup_{y>0} \left[\int |f(x + \mathrm{i}\, y)|^p \, dx \right]^{1/p} < \infty$$

- The definition by means of maximal functions works in all dimensions.

Definition of $H^p(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$

- $\mathbf{R}^2_+$ - the upper half plane.
- $\mathbf{R}^2$ is the boundary of $\mathbf{R}^2_+ \times \mathbf{R}^2_+$
- A point in $\mathbf{R}^2_+ \times \mathbf{R}^2_+$ will be denoted (t, y) where $t = (t_1, t_2) \in \mathbf{R}^2$ and $y = (y_1, y_2),\ y_i \geq 0$

Definition of $H^p(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$

- $\psi(t) \in C^1(\mathbf{R})$ is an even function supported on $[-1, 1]$ and $\int_{-1}^{1} \psi(t)dt = 0$
- $\psi_y(t) = (1/y)\psi(t/y)$ for $y > 0$
- $\psi_y(t) = \psi_{y_1}(t_1)\psi_{y_2}(t_2)$ for $t = (t_1, t_2) \in \mathbf{R}^2$ and $y = (y_1, y_2), \; y_i \geq 0$
- For $f : \mathbf{R}^2 \to \mathbf{R}$ define $f(t, y) = f * \psi_y(t)$

Definition of $H^p(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$

- If $x = (x_1, x_2) \in \mathbf{R}^2$, $\Gamma(x)$ will denote the product cone $\Gamma(x) = \Gamma(x_1) \times \Gamma(x_2)$ where

$$\Gamma(x_i) = \{(t_i, y_i) \in \mathbf{R}^2_+ : |x_i - t_i| < y_i\}, \;\; i = 1, 2$$

- Given a function f on $\mathbf{R}^2$ we define its double S-function by

$$S^2(f)(x) = \int\int_{\Gamma(x)} |f(t, y)|^2 \frac{dtdy}{y_1^2 y_2^2}.$$

- For $1 < p < \infty$ It's know that

$$\|S(f)\|_p \leq c_p \|f\|_p$$

Definition of $H^p(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$

- We define functions in $H^p(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$, $0 < p < \infty$ as those functions f with $S(f) \in L^p(\mathbf{R}^2)$
- $\|f\|_{H^p} = \|S(f)\|_p$
- This definition of H^p spaces is equivalent to the one defined via boundary values of functions of bi-holomorphic functions on $\mathbf{R}^2_+ \times \mathbf{R}^2_+$

Atomic Decomposition on $H^1(\mathbf{R})$

- **Definition** (on $\mathbf{R}$) An *atom* is a function $a(x)$ supported on an interval I such that

$$\int_I a(x)dx = 0 \quad \text{and} \quad \|a(x)\|_\infty \leq \frac{1}{|I|}$$

- **Theorem** (R. Coifman) $f \in H^1(\mathbf{R})$ if and only if f can be written as $f = \sum \lambda_k a_k$ where a_k are atoms and $\lambda_k \geq 0$ satisfy $\sum |\lambda_k| \leq A\|f\|_{H^1}$.

Atomic Decomposition on $H^1(\mathbf{R}_+^2 \times \mathbf{R}_+^2)$

Definition (on $\mathbf{R}_+^2 \times \mathbf{R}_+^2$) An *atom* is a function $a(x_1, x_2)$ defined on $\mathbf{R}^2$ whose support is contained in some open set Ω of finite measure such that

1. $\|a\|_2 \leq \frac{1}{|\Omega|^{1/2}}$
2. a can be further decomposed into *elementary particles* a_R as follows:

   (i) $a_R = \sum_R a_R$ where a_R is supported in the triple of distinct dyadic rectangles $R \subset \Omega$ (say $R = I \times J$)

   (ii) $\int_I a(x_1, \overline{x_2})dx_1 = \int_J a(\overline{x_1}, x_2)dx_2 = 0$ for each $\overline{x_1} \in I$, $\overline{x_2} \in J$

   (iii) a_R is C^1 with $\|a_R\|_\infty \leq d_R$,

$$\left\|\frac{\partial a_R}{\partial x_1}\right\|_\infty \leq \frac{d_R}{|I|}, \quad \left\|\frac{\partial a_R}{\partial x_2}\right\|_\infty \leq \frac{d_R}{|J|}$$

   with $\sum d_R^2 |R| \leq A/|\Omega|$

- **Theorem 1.** (Chang and Fefferman) $f \in H^1(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$ if and only if f can be written as $f = \sum \lambda_k a_k$ where a_k are atoms and $\lambda_k \geq 0$ satisfy $\sum \lambda_k \leq A\|f\|_{H^1}$.

- **Outline of the proof.** It's enough to show that if a is an atom then $a \in H_1$. Suppose a is an atom supported in the open set Ω, $a = \sum_{R \subset \Omega} a_R$.
  For each rectangle $S = I \times J$ let

$$S_+ = \left\{ (t, y) \in \mathbf{R}_+^2 \times \mathbf{R}_+^2;\ t \in S,\ \frac{|I|}{2} < y_1 \leq |I|,\ \frac{|J|}{2} < y_2 \leq |J| \right\}.$$

  For each point $x \in \mathbf{R}^2$, let S_x denote the collection of dyadic rectangles containing the point x. Then

$$\begin{aligned} S^2(a)(x) &= \int\int_{\Gamma(x)} |a(t,y)|^2 \frac{dtdy}{y_1^2 y_2^2} \\ &\lesssim \sum_{S \in S_x} \int\int_{S_+} |a(t,y)|^2 \frac{dtdy}{y_1^2 y_2^2} \end{aligned}$$

Atomic Decomposition on $H^1(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$

- $a(x) \in H^1$ iff $S(a) \in L^1$

$$\|S(a)\| = \int S(a)(x)dx$$

$$= \int_{\{M(1_\Omega)(x)<1/4\}} S(a)(x)dx + \int_{\{M(1_\Omega)(x)\geq 1/4\}} S(a)(x)dx.$$

- We have $|\{M(1_\Omega)(x) \geq 1/4\}| \leq c|\Omega|$

$$\int_{\{M(1_\Omega)(x)\geq 1/4\}} S(a)(x)dx \leq$$

$$\left(\int S^2(a)(x)dx\right)^{1/2} |\{M(1_\Omega)(x) \geq 1/4\}|^{1/2}$$

$$\leq c\|a\|_2|\Omega|^{1/2} \leq c.$$

- $\int_{\{M(1_\Omega)(x)<1/4\}} S(a)(x)dx < c$
- Hence $S(a) \in L^1$ implying $a(x) \in H^1$.

Calderon-Zygmund Decomposition

- **Theorem** (Calderon-Zygmund Decomposition). $f \in L^1(\mathbf{R}^n)$ and $\alpha > 0$ then there exists a disjoint collection of dyadic cubes $\{Q_i : i = 1, 2, ...\}$ such that

$$\alpha < \frac{1}{|Q_i|} \int_{Q_i} |f(x)| dx \leq 2^n \alpha, \quad i = 1, 2, ...,$$

  and

$$f(x) \leq \alpha \quad \text{for a.e. } x \in \mathbf{R}^n \setminus \cup_{i=1}^{\infty} Q_i.$$

- Given f as above, we can write f as the sum of a "good" function g and a "bad" function b, $f = g + b$, where $g \leq 2^n \alpha$ and b is supported on $\cup_{i=1}^{\infty} Q_i$ with

$$\int_{Q_i} |b(x)| dx \leq 2^n \alpha |Q_i| \quad \text{and} \quad \int_{Q_i} b(x) dx = 0.$$

Calderon-Zygmund Decomposition on Product Domains

- **Calderon-Zygmund Lemma** (Chang and Fefferman). Let $\alpha > 0$ be given and $f \in L^p(\mathbf{R}^2)$, $1 < p < \infty$. Then we may write $f = g + b$ where $g \in L^2(\mathbf{R}^2)$ and $b \in H^1(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$ with

$$\|g\|_2^2 \le \alpha^{2-p}\|f\|_p^p \quad \text{and} \quad \|b\|_{H^1} \le c\alpha^{1-p}\|f\|_p^p,$$

  where c is a universal constant.

- **Remark** It is shown that there exist constants λ_k and atoms b_k with $\sum |\lambda_k| \le \alpha^{1-p}\|f\|_p^p$ and $f = g + \sum \lambda_k b_k$.
- Theorem 1 implies that $b = \sum \lambda_k b_k$ is in H^1.

- **Theorem 2** (Chang and Fefferman). Let T be a linear operator which is bounded from $H^1(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$ to $L^1(\mathbf{R}^2)$ and bounded on $L^2(\mathbf{R}^2)$. Then T is bounded on $L^p(\mathbf{R}^2)$ for all $1 < p < 2$.
- **proof** Let $f \in L^p(\mathbf{R}^2)$ and $\alpha > 0$. According to the Calderon-Zygmund Lemma, we may write $f = g + b$ where

$$\|g\|_2^2 \leq \alpha^{2-p}\|f\|_p^p \quad \text{and} \quad \|b\|_{H^1} \leq c\alpha^{1-p}\|f\|_p^p$$

- $$m\{|Tf| > \alpha\} \leq m\{|Tg| > \alpha/2\} + m\{|Tb| > \alpha/2\}$$

$$\leq c\left(\frac{1}{\alpha^2}\|Tg\|_2^2 + \frac{1}{\alpha}\|Tb\|_1\right)$$

$$\leq c\left(\frac{1}{\alpha^2}\|g\|_2^2 + \frac{1}{\alpha}\|b\|_{H^1}\right)$$

Calderon-Zygmund Decomposition on Product Domains

$$m\{|Tf| > \alpha\} \leq m\{|Tg| > \alpha/2\} + m\{|Tb| > \alpha/2\}$$

$$\leq c\left(\frac{1}{\alpha^2}\|Tg\|_2^2 + \frac{1}{\alpha}\|Tb\|_1\right)$$

$$\leq c\left(\frac{1}{\alpha^2}\|g\|_2^2 + \frac{1}{\alpha}\|b\|_{H^1}\right)$$

$$\leq c\frac{1}{\alpha^p}\|f\|_p^p$$

- T is weak-type (p, p) for $1 < p < 2$.
- By the Marcinkiewicz Theorem T is bounded on L^p for $1 < p < 2$.

Atomic Decomposition on $H^p(\mathbf{R})$, $0 < p < 1$

- **Definition** (on $\mathbf{R}$) An *p-atom* is a function $a(x)$ supported on an interval I such that

$$\|a(x)\|_\infty \leq \frac{1}{|I|^{1/p}} \quad \text{and} \quad \int_I a(x) x^k dx = 0$$

for all $0 \leq k \leq \frac{1}{p} - 1$.

- **Theorem** (R. Coifman) $f \in H^p(\mathbf{R})$ if and only if f can be written as $f = \sum \lambda_k a_k$ where a_k are p-atoms and $\lambda_k \geq 0$ satisfy

$$A\|f\|_{H^p}^p \leq \sum |\lambda_k|^p \leq B\|f\|_{H^p}^p.$$

Definition (on $\mathbf{R}^2_+ \times \mathbf{R}^2_+$) An *p-atom* is a function $a(x_1, x_2)$ defined on $\mathbf{R}^2$ whose support is contained in some open set Ω of finite measure such that

1. $\|a\|_2^2 \leq |\Omega|^{1-2/p}$
2. a can be further decomposed into *elementary particles* a_R as follows:
   - (i) $a_R = \sum_R a_R$ where a_R is supported in the triple of distinct dyadic rectangles $R \subset \Omega$ (say $R = I \times J$)
   - (ii) $\int_I a(x_1, \overline{x_2}) x_1^k dx_1 = \int_J a(\overline{x_1}, x_2) x_2^k dx_2 = 0$ for each $\overline{x_1} \in I$, $\overline{x_2} \in J$ and $0 \leq k \leq k(p)$, where $k(p) = 2/p - 3/2$
   - (iii) a_R is C^m with $\|a_R\|_\infty \leq d_R$,

$$\left\| \frac{\partial^m a_R}{\partial x_1^m} \right\|_\infty \leq \frac{d_R}{|I|^m}, \quad \left\| \frac{\partial^m a_R}{\partial x_2^m} \right\|_\infty \leq \frac{d_R}{|J|^m}, \ 0 < m \leq k(p) + 1$$

with $\sum d_R^2 |R| \leq A|\Omega|^{1-2/p}$

- **Theorem 3.** (Chang and Fefferman) $f \in H^p(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$ then we may write $f = \sum \lambda_k a_k$ where a_k are p-atoms and $\lambda_k \geq 0$ satisfy $\sum \lambda_k^p \leq c_p \|f\|^p_{H^p}$.